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Optimal Guidance of Aeroassisted Transfer Vehicles Based on Matched Asymptotic Expansions

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In this paper we address and clarify a number of issues related to matched asymptotic expansion analysis of skip trajectories or any class of problems that give rise to inner layers that are not associated directly with satisfying boundary conditions. The procedure for matching inner and outer solutions and using the composite solution to satisfy boundary conditions is developed and rigorously followed to obtain a set of algebraic equations for the problem of inclination change with minimum energy loss. Repeated solution of these algebraic equations along the trajectory, treating each current state as an initial state, constitutes a feedback guidance algorithm. The solution is uniformly valid to zero order in an expansion parameter.

I. Introduction

An objective in any guidance study is the development of solutions that are implementible in real time and on board the vehicle. Therefore, solutions that are both near optimal and that require a minimum of computation are of primary interest. The ideal formulation from a computational point of view is one that reduces the solution of the governing differential equations to a set of algebraic equations, therefore eliminating the need for multiple shooting or quadrature. In this paper we address and clarify a number of issues related to matched asymptotic expansion (MAE) analysis of skip trajectories or any class of problems that give rise to inner layers that are not associated directly with satisfying boundary conditions. In addition, the procedure for matching inner and outer solutions and using the composite solution to satisfy boundary conditions is rigorously followed in developing a complete algebraic solution to the problem of inclination change with minimum energy loss.

Relation to Earlier Results

An extensive survey paper presented by Mease[1] gives the current status on the optimization of aeroassisted orbit transfer trajectories. In view of low-cost transportation being a key to the utilization and exploration of space, important issues such as payload mass delivery capability and aerodynamic heating considerations are discussed. Aeroassisted transfer trajectories give rise to a difficult optimization problem from a guidance point of view. In general, numerical methods are required for an exact solution. Examples of numerical solutions to optimal aeroassisted orbit transfer problems may be found in Refs. 2 and 3 and in earlier works cited in these references. In Ref. 2 a family of problems were studied in the context of optimal aeroassisted orbit transfer, including the minimization of the time integral of the flight path angle squared. The solution to this latter problem results in nearly grazing trajectories that take place in an altitude range where viscous effects are expected. It is shown that the nearly grazing solution is a useful engineering compromise between energy requirements and aerodynamic heating requirements. Optimization subject to a hard constraint on heating rate was considered in Ref. 3 by reformulation as a parameter optimization problem.

Approximation methods can be employed to obtain analytic solutions or to reduce the solution to a quadrature. However, it is difficult to precisely satisfy terminal constraints using these approximate forms, because the nature of aeroassisted transfers is that the controls (typically lift and bank angle) are most effective while the vehicle is well within the atmosphere. Adjustments near the end of the maneuver to reduce terminal errors rapidly lead to control saturation. References 4–6 typify the studies that have been performed on the problem of optimal aeroassisted orbit plane change and that are directed at obtaining analytical results. The author in Ref. 4 was able to integrate the state and costate equations by assuming that Loh's term, $M(h, V)$, is constant over the trajectory. The term $M(h, V)$ represents the sum of gravitational and inertial forces and is nearly zero for the entry phase of the maneuver but unfortunately undergoes a rapid variation during the exit phase. In Ref. 5 a regular perturbation method is used in which the perturbation parameter is identified as the ratio of the atmospheric scale height to the planet radius, and solutions are presented up to first order in the perturbation parameter. The solution approach requires that quadratures be performed at each control update for the first-order correction essential to account for variations in M near the end of the trajectory. The approach applied in Ref. 6 is similar to that given in Ref. 5 except for the use of an alternate independent variable. Unfortunately, large control effort can be observed near the end of the trajectory to satisfy terminal constraints. The interesting feature in Refs. 5 and 6 is that the zero-order solution corresponds to the constant M approximation in Ref. 4, for which complete analytic results are available.

The motivation for this paper is that, by its nature, the aeroassisted transfer problem is better suited to analysis by singular perturbations rather than by regular perturbation analysis. In fact, it will be shown that the regular perturbation problem formulated in Refs. 5 and 6 is actually the inner part of a two-time-scale analysis based on singular perturbation theory. The interesting feature here is that M variations are now accounted for in the zero-order outer problem (where the motion is Keplerian), for which an analytic solution can be obtained. The zero-order inner solution corresponds to the solution presented in Ref. 4, but with important differences that pertain to the treatment of boundary conditions in the MAE approach. The consequence of these results is that the problem can be reduced to a set of algebraic equations whose solution forms the basis for a feedback guidance algorithm. Moreover, it is anticipated that this approach (when extended to first-order analysis) will be more accurate in satisfying terminal constraints than the first-order solution obtained by the regular perturbation expansion in Refs. 5 and 6.

In Refs. 7 and 8, a MAE analysis is performed in which the perturbation parameter is the same as that used in Ref. 5. In Ref. 7 the expressions for the matching conditions are simplified over those obtained in earlier studies by using the inner solution alone to satisfy initial conditions. In Ref. 8 the state equations are integrated under the assumption of constant controls, and the expressions for the matching conditions are simplified by using the outer solution to satisfy initial conditions. This matching procedure is taken in Ref. 7 to obtain an optimal lift control solution of an atmospheric entry problem and in Ref. 8 to approximate an atmospheric skip

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trajectory for fixed controls. However, these approximate matching approaches are not recommended here for guidance law development since they are valid only when the initial condition lies either far outside or well inside the atmosphere.

Singular perturbation methods for re-entry and aeroassisted transfer trajectories have in more recent times been explored in Refs. 9–13. References 9–11 consider the re-entry problem. In Ref. 9 a singular perturbation parameter is artificially introduced on the left-hand sides of the altitude and flight path angle equations of motion. Several analytical guidance algorithms are derived for re-entry and evaluated by comparison to numerically obtained optimal solutions. Although this approach appears useful for re-entry problems, it does not yield a satisfactory solution for aeroassisted transfer problems because the boundary-layer dynamics associated with satisfying the terminal constraints are intractable. A suboptimal guidance algorithm is derived and evaluated in Ref. 10 that can be used in conjunction with the re-entry solutions in Ref. 9 to satisfy terminal constraints associated with aeroassisted transfer problems. In Ref. 11, a MAE analysis is performed in which the perturbation parameter is the same as that used in Ref. 5. The state equations are integrated under the assumption of constant controls, and the expressions for the matching conditions are simplified over those obtained in earlier studies by using the outer solution alone to satisfy initial conditions. Again, this approach is not recommended here for guidance law development since it can only be used as an approximation when the initial condition lies far outside the atmosphere.

The application of MAE to aeroassisted transfer trajectories has been addressed in Refs. 12 and 13. In Ref. 12, a general optimization problem is considered, and analytical results for the costate equations are presented. The problem is reduced to a set of constants of integration that are to be used to satisfy transversality conditions. Unfortunately, since the transversality conditions involve both states and costates, and since the state equations are not tractable in the inner layer, a multiple shooting method would have to be used to determine these parameters. Reference 13 considers the development of an atmospheric guidance law for planar skip trajectories in which the controls are treated as constants to be updated at each guidance computation. This reference also identifies several inconsistencies that were encountered in the analysis and in earlier studies and offers a way to partially correct for them.

MAE analysis of aeroassisted transfer problems is fundamentally different from re-entry problems in that the boundary conditions are given outside the region of singularity where the inner layer occurs. Aeroassisted transfer problems give rise to inner layers (regions where state and control variables can exhibit rapid variation) that correspond to intervals where the maneuver is dominated by aerodynamic forces. Thus the inner solution is associated with the aeroassisted portion of the maneuver. However, key issues that have been overlooked in previous studies on skip trajectories are 1) the role played by the inner solution in satisfying the boundary conditions, 2) the need for left and right outer solutions, 3) the role that the inner solution plays in joining the discontinuities that occur between the outer solutions through the matching conditions, and 4) the need to properly select the reference altitude used in defining the independent variable of integration in the outer solution.

Outline of Paper

Section II gives a general description of our approach in applying MAE to aeroassisted transfer trajectory analysis and establishes some notation to be followed throughout the paper. Section III uses the approach outlined in Sec. II in analyzing the orbit transfer problem of inclination change with minimum energy loss in the atmosphere. The analysis makes use of the problem formulation and analytic results in Ref. 12 for the outer solution. Then a transformation of variables is made to solve the inner problem using the analytical results presented in Ref. 4. An Appendix is included giving details pertaining to intermediate derivations.

II. Matched Asymptotic Analysis Procedure

In any aeroassisted transfer problem, the vehicle passes through two distinct regions in terms of the dominating forces (Figs. 1 and 2). In the high-altitude (outer) region, gravitational and inertial forces

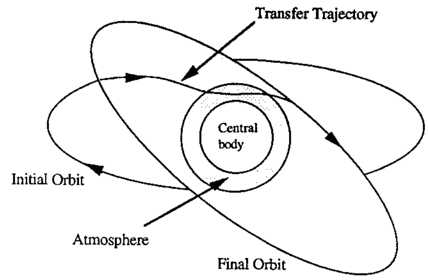

Fig. 1 Aeroassisted orbit transfer.

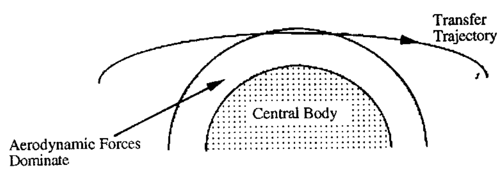

Fig. 2 Typical skip trajectory.

dominate and the motion can be approximated as Keplerian with the atmospheric effect considered as a perturbation. The low-altitude (inner) region is dominated by aerodynamic forces, with the gravitational and the inertial effects considered as perturbations. It is crucial to note that the initial (approaching) and final (retreating) parts of the transfer trajectory are distinctly different in their trajectory parameters (when approximated as Keplerian arcs) as a consequence of what occurs in the aerodynamically dominated region.

The method of MAE can be used as a mathematical realization of the above intuitive description. It decomposes the total problem into two simpler subproblems (known as the inner and outer problems), which are appropriate for the separate regions of the total trajectory. The process of matching the solutions of these subproblems and forming a composite solution accounts for the perturbing effects as well in a mathematically precise manner. For purposes of guidance law development the objective is to obtain approximate analytical solutions in the outer and inner regions separately and then to combine them to form a uniformly valid composite approximation for the entire maneuver.

To elaborate on this idea, consider the system of equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t, t/\varepsilon, \varepsilon) \tag{1}$$

The function f is assumed analytic in the region of interest with respect to its arguments, and in addition it is assumed to have the property that $\lim_{\varepsilon \to 0} f(x, t, t/\varepsilon, \varepsilon)$ exists for $t \neq 0$. The problem is singular in that the limit is not defined at $t = 0$. Conceptually, optimal-control problem formulations may appear in this form after eliminating the control using the optimality condition, so that only state and costate variables appear in the equations of motion. This results in a two-point boundary-value probem, with left and right boundary conditions of the form $\Psi_L(x_i, t_i) = 0$ and $\Psi_R(x_f, t_f) = 0$. Of particular interest here is the situation where $t_i < 0 < t_f$.

The solution of Eq. (1) is sought in the form of an asymptotic series in ε,

$$x^o(t, \varepsilon) = x_0^o(t) + \varepsilon x_1^o(t) + \varepsilon^2 x_2^o(t) + \cdots \tag{2}$$

which is referred to as the "outer" expansion. The leading term in Eq. (2) is obtained as the solution of Eq. (1) for $\varepsilon = 0$. However, due to the singularity in f at $t = 0$, it is not a uniformly valid $\mathcal{O}(\varepsilon)$ approximation of $x(t, \varepsilon)$. Note that for the situation $t_i < 0 < t_f$, two outer expansions are required: one for $t < 0$ and one for $t > 0$.

Only zero-order solutions are used in the analysis that follows, and to distinguish between the left and right zero-order outer solutions, we denote them by ${}^L x_0^o(t)$ and ${}^R x_0^o(t)$, where the superscript o denotes outer, the subscript 0 denotes zero order, and the superscripts L and R distinguish the left and right solutions. These solution segments are illustrated in Fig. 3, where it is important to note that in general there is a discontinuity at $t = 0$.

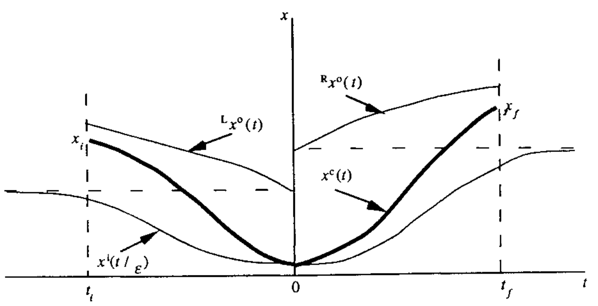

Fig. 3 Illustration of outer, inner, and composite solutions.

To examine the solution in the neighborhood of $t = 0$, Eq. (1) is expressed in terms of a stretched independent variable $\tau = t/\varepsilon$,

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \varepsilon f(x, \varepsilon\tau, \tau, \varepsilon) \tag{3}$$

The solution of Eq. (3) is sought in the form of an asymptotic series in ε,

$$x^i(\tau, \varepsilon) = x_0^i(\tau) + \varepsilon x_1^i(\tau) + \varepsilon^2 x_2^i(\tau) + \cdots \tag{4}$$

which is referred to as the "inner" expansion. Again, only the leading term in Eq. (4) is considered here.

Matching the inner solution to the outer solution is done separately for the left and right parts of the outer solution. To zero order in ε, this is accomplished by enforcing the following relations:

$$x_0^i(-\infty) = {}^L x_0^o(0), \qquad x_0^i(\infty) = {}^R x_0^o(0) \tag{5}$$

where $x_0^i(-\infty)$, ${}^L x_0^o(0)$, $x_0^i(\infty)$, and ${}^R x_0^o(0)$ are obtained by taking appropriate limits in their respective arguments τ and t. The consequence of Eq. (5) on the limit behavior of $x_0^i(\tau)$ is illustrated in Fig. 3, where the solution is shown superimposed with $x_0^o(t)$ in terms of the original independent variable t. The limit values in Eq. (5) also define the common parts of the inner and outer solutions.

At this stage a uniformly valid composite approximation can be constructed by the method of additive composition. The additive composition is obtained by taking the sum of the outer and inner solutions and subtracting the common part. In the left part of the trajectory, the composite solution is given by

$${}^L x_0^c(t, \varepsilon) = {}^L x_0^o(t) + x_0^i(t/\varepsilon) - {}^L x_0^o(0), \qquad t < 0 \tag{6}$$

and in the right part by

$${}^R x_0^c(t, \varepsilon) = {}^R x_0^o(t) + x_0^i(t/\varepsilon) - {}^R x_0^o(0), \qquad t > 0 \tag{7}$$

From Eqs. (6) and (7) it is seen that at $t = 0$ the composite solution takes the form

$${}^L x_0^c(0, \varepsilon) = {}^R x_0^c(0, \varepsilon) = x_0^i(0) \tag{8}$$

Thus the composite solution is continuous at $t = 0$.

From the above discussion it is apparent that the inner solution (and the subsequent matching and forming of the composite solution) can be viewed as a process whereby the discontinuity between ${}^L x_0^o(t)$ and ${}^R x_0^o(t)$ at $t = 0$ is taken into account. In a skip trajectory, this discontinuity is caused by the change that occurs in the trajectory parameters during the osculating atmospheric portion of the maneuver. The resulting composite solution is illustrated by the continuous bold line in Fig. 3.

From Fig. 3 it is also apparent that the composite solution must be used to satisfy the boundary conditions. In Refs. 11–13 only the outer solution was used to satisfy the bounadry conditions at $t = t_i$. In general, the contribution that $x_0^i(t/\varepsilon)$ makes to the boundary conditions depends on t_i and t_f. When the boundary conditions are far removed from the region of influence of the inner solution, the contribution may be small. However, in an optimization problem it is well known that solutions exhibit large sensitivity to the boundary values of the costate variables. Also, in the context of developing a guidance algorithm, the initial condition should always be viewed as occurring anywhere along the trajectory. In Ref. 13 use of the outer solution alone to satisfy initial conditions led to discrepancies in the matching conditions that could not be resolved in a straightforward way.

In the analysis that follows, we adopt the convention of using

$$t = h = (r - r_s)/r_s \tag{9}$$

where r_s is a reference radius. Although h is not monotonic, it is apparent from Fig. 3 that this presents no conceptual difficulty in the outer solution, since separate constants of integration are defined for entry and exit. It is however important to transform to a monotonic independent variable when integrating the inner dynamics. For this reason we employ the transformation used in Ref. 4. The left and right composite solutions in Eqs. (6) and (7) are then distinguished by the sign of the flight path angle.

Note that it is immediately obvious from Fig. 3 that $t = h = 0$ corresponds to $r_s = r_{\min}$, where $r_{\min}$ is the minimum radius over the trajectory. Thus, to determine r_s, it is essential to use the condition

$$\gamma_0^c(0, \varepsilon) = \gamma_0^i(0) = 0 \tag{10}$$

The approach followed in the next section is first to express the inner and outer solution in terms of constants of integration. The constants of integration for the left outer solution will be viewed as the variables used to ultimately satisfy the left and right boundary conditions. The constants of integration for the inner solution will be viewed as unknowns to be determined in terms of the left outer solution constants using the left matching condition in Eq. (5). The constants of integration for the right outer solution are in turn evaluated in terms of the inner solution constants using the right matching condition in Eq. (5). Finally, we would like to make note of the following relation, which holds when the jth component of x is constant in the outer solution:

$$x_{0j}^c(t, \varepsilon) = x_{0j}^i(t/\varepsilon) \tag{11}$$

which follows directly from Eqs. (6) and (7). This relation is used several times in the analysis that follows.

III. Inclination Change with Minimum Energy Loss

Problem Formulation

The three-dimensional point-mass equations of motion for a lifting vehicle over a spherical nonrotating planet are given by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = V \sin\gamma \tag{12}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{V\cos\gamma\cos\psi}{r\cos\varphi} \tag{13}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{V\cos\gamma\sin\psi}{r} \tag{14}$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{\rho s C_D V^2}{2m} - g\sin\gamma \tag{15}$$

$$V\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\rho s C_L V^2 \cos\mu}{2m} - \left(g - \frac{V^2}{r}\right)\cos\gamma \tag{16}$$

$$V\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\rho s C_L V^2 \sin\mu}{2m\cos\gamma} - \frac{V^2\cos\gamma\cos\psi\tan\varphi}{r} \tag{17}$$

For a Newtonian gravitational field we have

$$g(r) = g_s r_s^2 / r^2 \tag{18}$$

where the subscript s denotes a reference radius defined to be the minimum trajectory radius. An exponential atmosphere model is used:

$$\rho(r) = \rho_s e^{-\beta(r - r_s)}, \qquad \beta = 1/H_s \tag{19}$$

where H_s is the scale height. The lift and drag coefficients are assumed to be of the form

$$C_L = C_L^* \lambda, \qquad C_D = C_D^*(1+\lambda^2)/2 \tag{20}$$

where the constants C_L^* and C_D^* are the lift and drag coefficients corresponding to the maximum lift-to-drag ratio and λ is the normalized lift coefficient.

Defining the dimensionless quantities

$$h = (r - r_s)/r_s, \qquad u = V^2/g_s r_s, \qquad E^* = C_L^*/C_D^* \tag{21}$$

$$B = C_L^* \rho_s s / 2m\beta, \qquad \varepsilon = 1/\beta r_s = H_s/r_s \tag{22}$$

and using h in place of t as the independent variable, the state equations can be written as follows:

$$\frac{d\theta}{dh} = \frac{\cos\psi \cot\gamma}{(1+h)\cos\varphi} \tag{23}$$

$$\frac{d\varphi}{dh} = \frac{\sin\psi \cot\gamma}{(1+h)} \tag{24}$$

$$\frac{du}{dh} = -\frac{Bu(1+\lambda^2)e^{-h/\varepsilon}}{\varepsilon E^* \sin\gamma} - \frac{2}{(1+h)^2} \tag{25}$$

$$\frac{d\gamma}{dh} = \frac{B\lambda \cos\mu e^{-h/\varepsilon}}{\varepsilon \sin\gamma} + \left(\frac{1}{1+h} - \frac{1}{u(1+h)^2}\right)\cot\gamma \tag{26}$$

$$\frac{d\psi}{dh} = \frac{B\lambda \sin\mu e^{-h/\varepsilon}}{\varepsilon \sin\gamma \cos\gamma} - \frac{\cos\psi \tan\varphi \cot\gamma}{(1+h)} \tag{27}$$

The objective is to achieve an inclination change with minimum fuel consumption. For short-duration maneuvers, the change in the cross angle φ is small and the change in inclination is closely approximated by the change in the heading. Moreover, fuel consumption is nearly minimized by minimizing the energy loss in the atmospheric phase of the maneuver. Furthermore, in Ref. 14 it is shown that though the actual inclination change depends on the initial inclination, the starting point of the maneuver can be timed so as to obtain the maximum inclination change that is achieved when the initial inclination is zero. A consequence of this result is that the initial plane can be taken as the plane of reference, which we shall assume is the equatorial plane. Under these assumptions, $\theta(0)$ and $\varphi(0)$ can be set to zero without loss of generality and the heading angle ψ is used to approximate the change in the inclination angle. Thus θ and φ become ignorable coordinates, and the equations of motion reduce to

$$\frac{du}{dh} = -\frac{Bu(1+\lambda^2)e^{-h/\varepsilon}}{\varepsilon E^* \sin\gamma} - \frac{2}{(1+h)^2} \tag{28}$$

$$\frac{d\gamma}{dh} = \frac{B\lambda \cos\mu e^{-h/\varepsilon}}{\varepsilon \sin\gamma} + \left(\frac{1}{1+h} - \frac{1}{u(1+h)^2}\right)\cot\gamma \tag{29}$$

$$\frac{d\psi}{dh} = \frac{B\lambda \sin\mu e^{-h/\varepsilon}}{\varepsilon \sin\gamma \cos\gamma} \tag{30}$$

The parameter ε is the ratio of the atmospheric scale height to the minimum trajectory radius. In general, r_s is nearly equal to the planet's radius, and for the purpose of calculating ε to form the composite solution, it will be treated as such. For Earth the value of ε is of order 10^{-3}. The controls are the normalized lift coefficient λ and the bank angle μ.

Using the definition of u given in Eq. (22), the objective of minimizing energy loss can be equivalently expressed as

$$\max\{J\}, \qquad J = u_f \tag{31}$$

The Hamiltonian function associated with the state equations (28–30) has the form

$$\begin{aligned} H = {} & [-Bu(1+\lambda^2)e^{-h/\varepsilon}/\varepsilon E^* \sin\gamma - 2/(1+h)^2]P_u \\ & + \{B\lambda \cos\mu e^{-h/\varepsilon}/\varepsilon \sin\gamma + [1/(1+h) \\ & - 1/u(1+h)^2]\cot\gamma\}P_\gamma + [B\lambda \sin\mu e^{-h/\varepsilon}/\varepsilon \sin\gamma \cos\gamma]P_\psi \end{aligned} \tag{32}$$

where P_u, P_γ, and P_ψ are the associated costate variables. Assuming the controls λ and μ are not beyond there limits, their optimal value is obtained as a function of the state and costate variables by solving the optimality conditions $H_\lambda = 0$ and $H_\mu = 0$. The resulting expressions are

$$\begin{gathered} \lambda = E^*(P_\gamma \cos\mu + P_\psi \sin\mu/\cos\gamma)/2uP_u \\ \tan\mu = P_\psi/P_\gamma \cos\gamma \end{gathered} \tag{33}$$

Outer Solution[12]

The zero-order equations for the outer problem can be obtained by simply taking the limit as ε approaches zero on the right-hand side of Eqs. (28–30):

$$\frac{du_0^o}{dh} = -\frac{2}{(1+h)^2} \tag{34}$$

$$\frac{d\gamma_0^o}{dh} = \left(\frac{1}{1+h} - \frac{1}{u_0^o(1+h)^2}\right)\cot\gamma_0^o \tag{35}$$

$$\frac{d\psi_0^o}{dh} = 0 \tag{36}$$

The general solution for the outer system to zero order in ε is given by

$$u_0^o(h) = 2[c_1 + 1/(1+h)] \tag{37}$$

$$\cos\gamma_0^o(h) = c_2/(1+h)\sqrt{u_0^o(h)} \tag{38}$$

$$\psi_0^o(h) = c_3 \tag{39}$$

where c_1, c_2, and c_3 are constants of integration. The adjoint equations are given by

$$\frac{dP_x}{dh} = -H_x \tag{40}$$

where x is any of the state variables. In the outer region these equations to zero order in ε are

$$\frac{dP_{u0}^o}{dh} = -\frac{P_{\gamma 0}^o \cot\gamma_0^o}{\left[u_0^o(1+h)\right]^2} \tag{41}$$

$$\frac{dP_{\gamma 0}^o}{dh} = \frac{P_{\gamma 0}^o\left[1/(1+h) - 1/u_0^o(1+h)^2\right]}{\sin^2\gamma_0^o} \tag{42}$$

$$\frac{dP_{\psi 0}^o}{dh} = 0 \tag{43}$$

The solution to this system using Eqs. (37) and (38) is

$$P_{u0}^o(h) = -a_2/2u_0^o + a_1 \tag{44}$$

$$P_{\gamma 0}^o(h) = a_2 \tan\gamma_0^o \tag{45}$$

$$P_{\psi 0}^o(h) = a_3 \tag{46}$$

where a_1, a_2, and a_3 are constants of integration. Equations (37) and (38) provide the exact solution in the outer region and are the integrals of Keplerian motion that express conservation of energy and conservation of angular momentum.

Inner Solution[4]

In the inner region, where the aerodynamic force is dominant, a new stretched altitude is defined as

$$\eta = h/\varepsilon \tag{47}$$

However, to integrate the inner layer equations, we also adopt the transformations used in Ref. 4, which have the additional feature of transforming the independent variable from η to the monotonic variable ψ. Analytic solution also requires a small flight path angle approximation:

$$\cos\gamma \approx 1, \qquad \sin\gamma \approx \gamma \tag{48}$$

The following transformations are defined[15]:

$$w = Be^{-\eta}, \qquad v = E^* \ln(1/g_s r_s u) \tag{49}$$

The controls are also transformed to vertical and horizontal components:

$$\delta = \lambda\cos\mu, \qquad \sigma = \lambda\sin\mu \tag{50}$$

Invoking the above transformations, Eqs. (28–30) become

$$\frac{d\gamma}{d\psi} = \frac{\delta + \varepsilon\bar{M}(\varepsilon)}{\sigma} \tag{51}$$

$$\frac{dw}{d\psi} = -\frac{\gamma}{\sigma} \tag{52}$$

$$\frac{dv}{d\psi} = \frac{1+\delta^2+\sigma^2+\varepsilon G(\varepsilon)}{\sigma} \tag{53}$$

where $\varepsilon\bar{M}(\varepsilon)$ is Loh's term, which accounts for the gravitational and inertial effects on the motion. For $\bar{M}$ and G we have

$$\bar{M}(\varepsilon) = \{1 - e^{v/E^*} g_s r_s/[1+\varepsilon\ln(B/w)]\}/w[1+\varepsilon\ln(B/w)] \tag{54}$$

$$G(\varepsilon) = e^{v/E^*} g_s r_s E^* \sin 2\gamma / w[1+\varepsilon\ln(B/w)]^2 \tag{55}$$

The approximations used in Ref. 4 to integrate the state and costate equations were that $\varepsilon\bar{M}(\varepsilon)$ is constant over the trajectory and the term $\varepsilon G(\varepsilon)$ was neglected. We note here that setting $\varepsilon = 0$ in Eqs. (51) and (53) to obtain the zero-order inner equations corresponds to the approximations in Ref. 4 with $\varepsilon\bar{M}(\varepsilon) = 0$. In actuality, $\varepsilon\bar{M}(\varepsilon)$ is nearly zero during entry but undergoes a sharp variation during the exit phase. This is the main shortcoming to the approximation in Ref. 4. The interesting feature in this analysis is that the zero-order inner solution corresponds to the $\bar{M} = 0$ solution in Ref. 4 but that $\bar{M}$ variations are accounted for in the outer solution. It should also be pointed out that whereas the integrated solution bears a close resemblance to that in Ref. 4, it is totally different in the method of evaluating the constants of integration.

The transformed zero-order state equation in the inner region will be obtained by letting $\varepsilon = 0$ in Eqs. (51) and (53). The zero-order state equations for γ^i and v^i become

$$\frac{d\gamma_0^i}{d\psi_0^i} = \frac{\delta}{\sigma} \tag{56}$$

$$\frac{dv_0^i}{d\psi_0^i} = \frac{1+\delta^2+\sigma^2}{\sigma} \tag{57}$$

Using Eq. (50), the performance index in Eq. (31), expressed in the transformed variables, takes the form

$$\max\{J\}, \qquad J = e^{-v_f/E^*}/g_s r_s \tag{58}$$

Note that u_f in Eq. (31) and v_f in Eq. (58) represent the composite solution of the transformed final velocity to zero order and not the inner or outer solutions alone. The zero-order Hamiltonian function in the inner region is given by

$$H_0^i = P_{0\gamma}^i\delta/\sigma - P_{0w}^i\gamma^i/\sigma + P_{0v}^i(1+\delta^2+\sigma^2)/\sigma \tag{59}$$

and is constant since it does not depend on the independent variable ψ_0^i. The control functions are obtained from the optimality conditions $H_{0\delta}^i = 0$ and $H_{0\sigma}^i = 0$. They are given by

$$\delta = -P_{0\gamma}^i/2P_{0v}^i, \qquad \sigma^2 = \left(-P_{0w}^i\gamma^i - P_{0\gamma}^{i2}/4P_{0v}^i\right)/P_{0v}^i + 1 \tag{60}$$

The corresponding transformed zero-order costate equations in the inner region are obtained using Eq. (40):

$$\frac{dP_{0\gamma}^i}{d\psi_0^i} = \frac{P_{0w}^i}{\sigma} \tag{61}$$

$$\frac{dP_{0w}^i}{d\psi_0^i} = 0 \tag{62}$$

$$\frac{dP_{0v}^i}{d\psi_0^i} = 0 \tag{63}$$

In the Appendix it is shown that $H_0^i = 2\sigma P_{0v}^i$ and a consequence of this result and Eq. (63) is that σ is constant too.

At this stage the state equations (52), (56), and (57) and the costate equations (61–63) can be integrated with respect to ψ_0^i to give

$$\gamma_0^i(\psi_0^i) = -k_1\psi_0^{i2}/2 + k_2\psi_0^i + k_3 \tag{64}$$

$$w_0^i(\psi_0^i) = \left[k_1\psi_0^{i3}/6 - k_2\psi_0^{i2}/2 - k_3\psi_0^i + k_4\right]/\sigma \tag{65}$$

$$v_0^i(\psi_0^i) = (\sigma + 1/\sigma)\psi_0^i + \sigma\left[\left(\psi_0^i k_1 - k_2\right)^3\right]/3k_1 + k_5 \tag{66}$$

and

$$P_{0\gamma}^i(\psi_0^i) = \left(P_{0w}^i/\sigma\right)\psi_0^i + c \tag{67}$$

$$P_{0w}^i = \text{const}, \qquad P_{0v}^i = \text{const} \tag{68}$$

where k_3, k_4, k_5 and c, P_{0w}^i, P_{0v}^i are the constants of integration of the state and costate equations, respectively. The constants k_1 and k_2 are defined in terms of these constants as

$$k_1 = P_{0w}^i/2\sigma^2 P_{0v}^i, \qquad k_2 = -c/2\sigma P_{0v}^i \tag{69}$$

Matching Conditions

The method of additive composition (as described in Sec. II) is used to combine the zero-order inner and outer solutions into a single uniformly valid approximation. The additive composition is obtained by taking the sum of the solutions in the different regions and subtracting the common part. Matching implies agreement between the outer solution for small values of $h(h \to 0)$ and the inner solution for large values of $\eta(\eta \to \infty)$. Since the inner solution lies between two discontinuous outer solutions, matching is done separately in the left and right parts of the transfer trajectory, and two sets of matching conditions result. Each set of conditions involves matching of both the state and costate variables.

State Matching Conditions

The solution for the states in the outer region is given by Eqs. (37–39). Taking the limit $h \to 0$, we have

$$^L u_0^o(0) = 2\left(c_1^L + 1\right) \tag{70}$$

$$\cos\left[^L\gamma_0^o(0)\right] = c_2^L/\left[2\left(c_1^L+1\right)\right]^{\frac{1}{2}} \tag{71}$$

$$^L\psi_0^o(0) = c_3^L \tag{72}$$

where superscript L denotes the constants of integration for the left outer solution.

The solutions for the transformed inner variables are expressed with ψ_0^i as the independent variable. Conceptually, it is possible to perform an inverse transformation to express the inner solution in the original variables with η as the independent variable. However, the inverse transformation can be bypassed by first recognizing from Eq. (49) that

$$^{L}w_0^i(\infty) = 0 \tag{73}$$

$$^{L}v_0^i(\infty) = E^*\ell n\left[1/g_s r_s {}^{L}u_0^i(\infty)\right] = E^*\ell n\left[1/2g_s r_s\left(c_1^L + 1\right)\right] \tag{74}$$

where the second relation in Eq. (74) follows from Eq. (70) and enforcement of the matching condition $^{L}u_0^o(0) = {}^{L}u_o^i(\infty)$. Applying the matching condition to γ_0^i and ψ_0^i, we have

$$^{L}\gamma_0^i(\infty) = {}^{L}\gamma_0^o(0) = \cos^{-1}\left\{c_2^L/\left[2\left(c_1^L+1\right)\right]^{\frac{1}{2}}\right\} \tag{75}$$

$$^{L}\psi_0^i(\infty) = {}^{L}\psi_0^o(0) = c_3^L \tag{76}$$

Using Eqs. (73–76) to evaluate Eqs. (64–66) at $\eta = \infty$, we have

$$\cos^{-1}\left\{c_2^L/\left[2\left(c_1^L+1\right)\right]^{\frac{1}{2}}\right\} = -k_1 c_3^{L^2}/2 + k_2 c_3^L + k_3 \tag{77}$$

$$0 = k_1 c_3^{L^3}/6 - k_2 c_3^{L_2}/2 - k_3 c_3^L + k_4 \tag{78}$$

$$E^*\ell n\left[1/2g_s r_s\left(c_1^L+1\right)\right]$$

$$= (\sigma + 1/\sigma)c_3^L + \sigma\left[\left(c_3^L k_1 - k_2\right)^3\right]/3k_1 + k_5 \tag{79}$$

Equations (77–79) can be used to evaluate k_3, k_4, and k_5 in terms of the constants of integration for the state variables in the left-side outer solution. Recall that k_1 and k_2 have been defined in Eq. (69).

An exactly symmetric set of equations results from the matching conditions for the state variables on the right side:

$$\cos^{-1}\left\{c_2^R/\left[2\left(c_1^R+1\right)\right]^{\frac{1}{2}}\right\} = -k_1 c_3^{R^2}/2 + k_2 c_3^R + k_3 \tag{80}$$

$$0 = k_1 c_3^{R^3}/6 - k_2 c_3^{R^2}/2 - k_3 c_3^R + k_4 \tag{81}$$

$$E^*\ell n\left[1/2g_s r_s\left(c_1^R+1\right)\right]$$

$$= (\sigma + 1/\sigma)c_3^R + \sigma\left[\left(c_3^R k_1 - k_2\right)^3\right]/3k_1 + k_5 \tag{82}$$

Equations (80–82) relate c_1^R, c_2^R, and c_3^R to the constants of integration for the state variables in the inner solution.

Costate Matching Conditions

The left and right matching conditions for the costate variables are defined below. The inner solution for the costate variables corresponding to the transformed state variables is given in Eqs. (67–68) in terms of the constants of integration c, P_{0w}^i, and P_{0v}^i. To perform the matching with the outer costate solutions given in Eqs. (44–46), it is necessary to first find the corresponding expressions for P_{0u}^i and $P_{0\psi}^i$ for the inner solution. In effect, it amounts to transforming the inner solution back to the original problem variables.

The value of any costate variable at time t can be interpreted as the sensitivity of J to perturbations in the corresponding state variable at time t.[12] Thus we can write

$$P_u = \frac{\partial J}{\partial u}, \qquad P_v = \frac{\partial J}{\partial v} \tag{83}$$

From Eq. (50) we have

$$\frac{\partial v^i}{\partial u^i} = -\frac{E^*}{u^i} \tag{84}$$

Combining Eqs. (83) and (84), it becomes apparent that

$$P_{0u}^i = P_{0v}^i\frac{\partial v^i}{\partial u^i} = -\frac{E^* P_{0v}^i}{u_0^i} \tag{85}$$

To determine $P_{0\psi}^i$, we note that ψ_0^i is the independent variable in the transformed inner problem. Therefore, making use of the Hamilton–Jacobi–Belman equation,[12] we have

$$P_{0\psi}^i = \frac{\partial J}{\partial \psi_0^i} = -H_0^i \tag{86}$$

In the Appendix it is shown that $H_0^i = 2\sigma P_{0v}^i$; thus

$$P_{0\psi}^i = -2\sigma P_{0v}^i \tag{87}$$

Using Eqs. (85) and (87), we are prepared to carry out the costate matching process. Taking the limit $h \to 0$ in Eqs. (44–46) and using Eqs. (70–72), we have the left-side matching conditions

$$^{L}P_{0u}^o(0) = -a_2^L/4\left(c_1^L+1\right) + a_2^L = {}^{L}P_{0u}^i(\infty) \tag{88}$$

$$^{L}P_{0\gamma}^o(0) = a_2^L\tan\left(\cos^{-1}\left\{c_2^L/\left[2\left(c_1^L+1\right)\right]^{\frac{1}{2}}\right\}\right) = {}^{L}P_{0\gamma}^i(\infty) \tag{89}$$

$$^{L}P_{0\psi}^o(0) = a_3^L = {}^{L}P_{0\psi}^i(\infty) \tag{90}$$

Taking the limit $n \to \infty$ in Eqs. (67), (85), and (87) and making use of Eqs. (70), (72), and (76) and of the state matching condition $^{L}u_0^o(0) = {}^{L}u_0^i(\infty)$, we have

$$^{L}P_{0u}^i(\infty) = -E^* P_{0v}^i/2\left(c_1^L+1\right) \tag{91}$$

$$^{L}P_{0\gamma}^i(\infty) = \left(P_{0w}^i/\sigma\right)c_3^L + c \tag{92}$$

$$^{L}P_{0\psi}^i(\infty) = -2\sigma P_{0v}^i \tag{93}$$

Substituting Eqs. (91–93) in (88–90) yields

$$-a_2^L/4\left(c_1^L+1\right) + a_2^L = -E^* P_{0v}^i/2\left(c_1^L+1\right) \tag{94}$$

$$a_2^L\tan\left(\cos^{-1}\left\{c_2^L/\left[2\left(c_1^L+1\right)\right]^{\frac{1}{2}}\right\}\right) = \left(P_{0w}^i/\sigma\right)c_3^L + c \tag{95}$$

$$a_3^L = -2\sigma P_{0v}^i \tag{96}$$

Finally, in the Appendix it is shown that σ can be written as

$$\sigma = 1/\left(1 + k_2^2 + 2k_1k_3\right)^{\frac{1}{2}} \tag{97}$$

Equations (94) and (95) relate c, P_{0w}^i, and P_{0v}^i to the constants of integration for the left outer costate solution.

An exactly symmetric set of equations results from the matching conditions for the costate variables on the right side:

$$-a_2^R/4\left(c_1^R+1\right) + a_2^R = -E^* P_{0v}^i/2\left(c_1^R+1\right) \tag{98}$$

$$a_2^R\tan\left(\cos^{-1}\left\{c_2^R/\left[2\left(c_1^R+1\right)\right]^{\frac{1}{2}}\right\}\right) = \left(P_{0w}^i/\sigma\right)c_3^R + c \tag{99}$$

$$a_3^R = -2\sigma P_{0v}^i \tag{100}$$

Equations (88–100) relate a_1^R, a_2^R, and a_3^R to the constants of integration for the inner costate solution.

Composite Solution

The composite solution is constructed by the method of additive composition as outlined in Sec. II. For the state variables the form of the composite solution is

$$u_0^c(h) = u_0^o(h) + u_0^i(h/\varepsilon) - u_0^o(0) \tag{101}$$

$$\gamma_0^c(h) = \gamma_0^o(h) + \gamma_0^i(h/\varepsilon) - \gamma_0^o(0) \tag{102}$$

$$\psi_0^c(h) = \psi_0^o(h) + \psi_0^i(h/\varepsilon) - \psi_0^o(0) \tag{103}$$

where now $\varepsilon = H_s/\bar{r}_s$ and $\bar{r}_s$ is the planet radius [see the comment following Eq. (30)]. The solution for the outer state variables is given in Eqs. (37–39) and for the inner state variables in Eqs. (64–66). The outer left solution evaluated at $h = 0$ is given in Eqs. (70–72). Using Eq. (50), the left composite solution is given by

$$u_0^c(h) = 2/(1+h) + e^{-v_0^i(h/\varepsilon)/E^*}/g_s r_s - 2 \tag{104}$$

$$\gamma_0^c(h) = \cos^{-1}\left(c_2^L/(1+h)\left\{2\left[c_1^L + 1/(1+h)\right]\right\}^{\frac{1}{2}}\right) + \left\{-k_1\left[\psi_0^i(h/\varepsilon)\right]^2/2 + k_2\psi_0^i(h/\varepsilon) + k_3\right\} - \cos^{-1}\left\{c_2^L/\left[2\left(c_1^L+1\right)\right]^{\frac{1}{2}}\right\} \tag{105}$$

$$\psi_0^c(h) = \psi_0^i(h/\varepsilon) \tag{106}$$

Note that the composite solution for ψ is simply the inner solution since the outer solution is constant (see the general comment at the end of Sec. II). Since no outer constants of integration appear explicitly in the left composite solutions of u and of ψ, the left and right composite solutions are identical. The right composite solution for γ is given by

$$\gamma_0^c(h) = \cos^{-1}\left(c_2^R/(1+h)\left\{2\left[c_1^R + 1/(1+h)\right]\right\}^{\frac{1}{2}}\right) + \left\{-k_1\left[\psi_0^i(h/\varepsilon)\right]^2/2 + k_2\psi_0^i(h/\varepsilon) + k_3\right\} - \cos^{-1}\left\{c_2^R/\left[2\left(c_1^R+1\right)\right]^{\frac{1}{2}}\right\} \tag{107}$$

By Eq. (49), w_0^i as a function of h/ε is given as

$$w_0^i(h/\varepsilon) = Be^{-h/\varepsilon} \tag{108}$$

and Eq. (65) is used to solve for $\psi_0^i(h/\varepsilon)$ in terms of the altitude h and the constants of integration. Equation (66) is then used to evaluate $v_0^i(h/\varepsilon)$. Thus Eqs. (104–107) together with Eqs. (66), (66), and (108) provide the composite solution for the state variables as a function of h.

The form of the composite solution of the costate variables is

$$P_{0u}^c(h) = P_{0u}^o(h) + P_{0u}^i(h/\varepsilon) - P_{0u}^o(0) \tag{109}$$

$$P_{0\gamma}^c(h) = P_{0\gamma}^o(h) + P_{0\gamma}^i(h/\varepsilon) - P_{0\gamma}^o(0) \tag{110}$$

$$P_{0\psi}^c(h) = P_{0\psi}^o(h) + P_{0\psi}^i(h/\varepsilon) - P_{0\psi}^o(0) \tag{111}$$

and it is constructed the same way the composite solution for the states was obtained. The solution for the outer costate variables is given in Eqs. (44–46) and for the inner costate variables in Eqs. (67) and (68). The outer left solution evaluated at $h = 0$ is given in Eqs. (88–90). Using Eqs. (50), (85), and (87), the left composite solution is given by

$${}^LP_{0u}^c(h) = -a_2^L/4\left[c_1^L + 1/(1+h)\right] - P_{0v}^i E^*/\left(e^{v_0^i(h/\varepsilon)/E^*}/g_s r_s\right) + a_2^L/4\left(c_1^L+1\right) \tag{112}$$

$${}^LP_{0\gamma}^c(h) = a_2^L \tan\left\{\cos^{-1}\left(c_2^L/(1+h)2^{\frac{1}{2}}\left[c_1^L + 1/(1+h)\right]^{\frac{1}{2}}\right)\right\} + \left(P_{0w}^i/\sigma\right)\psi_0^i(h/\varepsilon) + c - a_2^L \tan\left\{\cos^{-1}\left[c_2^L/2^{\frac{1}{2}}\left(c_1^L+1\right)^{\frac{1}{2}}\right]\right\} \tag{113}$$

$${}^LP_{0\psi}^c(h) = a_3^L \tag{114}$$

The values of $v_0^i(h/\varepsilon)$ and $\psi_0^i(h/\varepsilon)$ are found from Eqs. (65), (66), and (108). Equations (46) and (87) for $P_{0\psi}$ show that it is constant in the outer and inner regions. Hence the composite solution for $P_{0\psi}$ is also constant and is simply the outer costate constant of integration. On the right side the composite solution takes the form

$${}^RP_{0u}^c(h) = -a_2^R/4\left[c_1^R + 1/(1+h)\right] - P_{0v}^i E^*/\left(e^{-v_0^i(h/\varepsilon)/E^*}/g_s r_s\right) + a_2^R/4\left(c_1^R+1\right) \tag{115}$$

$${}^RP_{0\gamma}^c(h) = a_2^R \tan\left(\cos^{-1}\left\{c_2^R/(1+h)2^{\frac{1}{2}}\left[c_1^R + 1/(1+h)\right]^{\frac{1}{2}}\right\}\right) + \left(P_{0w}^i/\sigma\right)\psi_0^i(h/\varepsilon) + c - a_2^R \tan\left\{\cos^{-1}\left[c_2^R/2^{\frac{1}{2}}\left(c_1^R+1\right)^{\frac{1}{2}}\right]\right\} \tag{116}$$

$${}^RP_{0\psi}^c(h) = a_3^R \tag{117}$$

From Eqs. (96) and (100) it is seen that $a_3^L = a_3^R$; thus the composite solution for $P_{0\psi}$ is constant all along the transfer trajectory. Equations (112–117) provide the composite solution for the costate variables as a function of h.

The reference radius r_s is also treated as an unknown parameter in the problem. As discussed in Sec. II, it is the distance to the lowest point of the transfer trajectory at which $h = 0$. Using the fact that at this point the composite solution for γ is zero provides the relation needed to evaluate r_s. The form of the composite solution for γ is given in Eq. (104). At $h = 0$ it becomes the inner solution evaluated at $h = 0$,

$$\gamma_0^c(0) = \gamma_0^i(0) \tag{118}$$

The solution for γ_0^i is given in Eq. (64). Equating to zero gives a quadratic relation for the value of ψ_0^i corresponding to $h = 0$ that has the following solution:

$$\psi_0^i(0) = k_2/k_1\left[1 \pm (1 + 2k_1k_3/k_2)^{\frac{1}{2}}\right] \tag{119}$$

This value of $\psi_0^i(0)$ is used in Eq. (65) to evaluate w_0^i at $h = 0$:

$$w_0^i(0) = \left[k_1\psi_0^i(0)^3/6 - k_2\psi_0^i(0)^2/2 - k_3\psi_0^i(0) + k_4\right]/\sigma \tag{120}$$

Finally, use of Eqs. (120) and (22) in Eq. (108) for $h = 0$ gives

$$w_0^i(0) = C_L^*\rho_s s/2m\beta \tag{121}$$

from which ρ_s (and thus r_s) can be found in terms of the constants of integration of the problem.

Enforcing Boundary Conditions

To complete the solution of the problem, the initial conditions and transversality conditions at the end point must be satisfied. As discussed in Sec. II, this is done using the composite solution and can be thought of as the process by which the constants of integration for the left outer solution are evaluated.

Assume that the initial conditions u_i, γ_i, and ψ_i are given, along with the initial radius r_i. For the purpose of exposition, assume that γ_f, ψ_f, and the final radius r_f are given. The corresponding transversality condition is

$$P_u^c(h_f) = 1 \tag{122}$$

where $h_i = (r_i - r_s)/r_s$ and $h_f = (r_f - r_s)/r_s$. Using u_i, γ_i, ψ_i, ψ_f, γ_f, and Eq. (122), the following equations result from enforcing the boundary conditions:

$$u_i = 2/(1+h_i) + e^{-v_0^i(h_i/\varepsilon)/E^*}/g_s r_s - 2 \tag{123}$$

$$\gamma_i = \cos^{-1}\left(c_2^L/(1+h_i)\left\{2\left[c_1^L + 1/(1+h_i)\right]\right\}^{\frac{1}{2}}\right) + \left\{-k_1\left[\psi_0^i(h_i/\varepsilon)\right]^2/2 + k_2\psi_0^i(h_i/\varepsilon) + k_3\right\} - \cos^{-1}\left\{c_2^L/\left[2\left(c_1^L+1\right)\right]^{\frac{1}{2}}\right\} \tag{124}$$

$$\psi_i = \psi_0^i(h_i/\varepsilon) \tag{125}$$

$$\psi_f = \psi_0^i(h_f/\varepsilon) \tag{126}$$

$$1 = -a_2^R/4\left[c_1^R + 1/(1+h_f)\right] - P_{0v}^i E^*/\left(e^{-v_0^i(h_f/\varepsilon)/E^*}/g_s r_s\right) + a_2^R/4\left(c_1^R+1\right) \tag{127}$$

$$\gamma_f = \cos^{-1}\left(c_2^R/(1+h_f)\left\{2\left[c_1^R + 1/(1+h_f)\right]\right\}^{\frac{1}{2}}\right) + \left\{-k_1\left[\psi_0^i(h_f/\varepsilon)\right]^2/2 + k_2\psi_0^i(h_f/\varepsilon) + k_3\right\} - \cos^{-1}\left\{c_2^R/\left[2\left(c_1^R+1\right)\right]^{\frac{1}{2}}\right\} \tag{128}$$

where h_i and h_f are defined below Eq. (122). To summarize, Eqs. (77–82), (94–100), (121), and (123–128) constitute a set of 20 equations for the 18 unknown constants of integration $c_1^L, c_2^L, c_3^L, k_3, k_4, k_5, c_1^R, c_2^R, c_3^R, a_1^L, a_2^L, a_3^L, c, P_{0w}^i, P_{0v}^i, a_1^R, a_2^R, a_3^R$ and for the parameters σ and r_s.

Guidance Law Implementation

The optimal-control expressions are given in Eq. (33). Assuming that the states are available for feedback, only estimates of the costates are required at each control computation along the trajectory. Feedback implementation entails treating the current state at each control update as a new initial state and calculating the costate values corresponding to the same time instant. The estimate for these costates (to zero order) are obtained by repetitively solving the 20 algebraic equations summarized at the end of Sec. III and then evaluating the costate values at $h = h_i$ using Eqs. (112–114).

During the entry phase, the complete system of equations is used as developed in Sec. III. During the exit phase, the left outer solution is discarded, and matching is required only between the right outer solution and the inner solution. In this case, the constants of integration for the inner solution are viewed as free parameters used to satisfy the boundary conditions.

Numerical results may be found in Ref. 16.

Higher Order Solutions

In Refs. 12 and 13 and in earlier references it is stated that the higher order left outer solutions for the state variables are zero. This is a consequence of using only the outer solution to satisfy the initial conditions. In fact they are not constant if the composite solution is used to satisfy the boundary conditions, which is the case in the present work.

In Sec. III it was shown that the inner problem is identical to the problem in Refs. 5 and 6 using a regular expansion method. Hence, it is apparent that all the higher order inner solutions will require the use of quadratures. Reference 17 develops a general procedure for constructing a matched asymptotic expansion of the Hamilton–Jacobi–Bellman (HJB) equation, analogous to the development in Refs. 5 and 6 for a regular expansion. It is shown that the higher order outer expansion terms are zero and that only an inner expansion is required.

An alternative approach to obtaining higher order corrections is to continue the expansion of the Euler system of equations. The higher order equations for the states and costates are coupled, linear, and inhomogeneous, and the contribution of higher order outer terms to the composite solution is not zero. The solution of these equations requires calculation of the state transition matrix, which can be derived analytically using the analytical zero-order solution and performing of a quadrature on both state and costate equations. This method has an important potential advantage over the former one in that it may be possible to fix the zero-order solution and precompute and store the quadratures along the zero-order solution as a function of a monotonic variable (such as total energy). Then at each control update, the state perturbations from the zero-order solution are accounted for in the first-order correction by treating them as initial conditions for the first-order solution.[18] This is not possible in an HJB expansion, since the first-order correction is valid only for the current values of the state and the independent variable and the quadrature must be repeated at each control update. It has been shown that, in the case of a regular perturbation, expansion of the HJB equation is otherwise equivalent to expansion of the Euler system of equations.[19]

IV. Conclusions

This paper has addressed and clarified a number of issues related to MAE analysis of skip trajectories. Application to the problem of inclination change with minimum energy loss has resulted in a zero-order solution in the form of a set of 20 algebraic equations. Repeated solution of these algebraic equations along the trajectory, treating each current state as an initial state, constitutes a feedback guidance algorithm.

Appendix: Auxiliary Derivations

Expressed in the transformed variables in the inner region, the Hamiltonian function has the form given in Eq. (59):

$$H_0^i = P_{0\gamma}^i \delta/\sigma - P_{0w}^i \gamma^i/\sigma + P_{0v}^i(1 + \delta^2 + \sigma^2)/\sigma \tag{A1}$$

The control functions are obtained from the optimality conditions $H_{0\delta}^i = 0$ and $H_{0\sigma}^i = 0$. Using Eq. (A1), after some manipulation we have

$$\delta = -P_{0\gamma}^i / 2P_{0v}^i, \qquad \sigma^2 = \left(-P_{0w}^i \gamma_0^i - P_{0\gamma}^{i^2}/4P_{0v}^i\right)/P_{0v}^i + 1 \tag{A2}$$

Using Eq. (A2) in Eq. (A1) yields

$$H_0^i = \left(2P_{0v}^i/\sigma\right)\left[\left(-P_{0w}^i\gamma_0^i - P_{0\gamma}^{i^2}/4P_{0v}^i\right)/P_{0v}^i + 1\right] = 2\sigma P_{0v}^i \tag{A3}$$

Since H_0^i and P_{0v}^i are constant in this problem, so is σ. Thus it should be possible to express σ in terms of constant parameters only, which is not immediately recognized in Eq. (A3) since it contains the functions $P_{0\gamma}^i$ and γ_0^i. However, use of Eq. (67) in Eq. (A3) gives

$$\sigma^2 = \left\{\left[\left(P_{0w}^i/\sigma\right)\psi_0^i + c\right]/2P_{0v}^i\right\}^2 - P_{0w}^i\gamma_0^i/P_{0v}^i + 1 \tag{A4}$$

From Eqs. (69), P_{0w}^i and c can be written as

$$P_{0w}^i = 2\sigma^2 P_{0v}^i k_1, \qquad c = -2\sigma P_{0v}^i k_2 \tag{A5}$$

Using these expressions and Eq. (64) in Eq. (A4), σ^2 becomes

$$\sigma^2 = 1/\left(1 + k_2^2 + 2k_1k_3\right) \tag{A6}$$

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